46(3) 2023

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Published online: 17 Feb 2022.

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LINEAR EQUIVALENCE OF (PSEUDO)COMPACT SPACES

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ABSTRACT. Given Tychonoff spaces X and Y, Uspenskij proved in [15] that if X is pseudocompact and $C_p(X)$ is uniformly homeomorphic to $C_p(Y)$, then Y is also pseudocompact. In particular, if $C_p(X)$ is linearly homeomorphic to $C_p(Y)$, then X is pseudocompact if and only if so is Y. This easily implies Arhangel'skii's theorem [1] which states that, in the case when $C_p(X)$ is linearly homeomorphic to $C_p(Y)$, the space X is compact if and only if Y is compact. We will establish that existence of a linear homeomorphism between the spaces $C_p^*(X)$ and $C_p^*(Y)$ implies that X is (pseudo)compact if and only if so is Y. We will also show that the methods of proof used by Arhangel'skii and Uspenskij do not work in our case.

1. Introduction.   All spaces in this article are assumed to be Tychonoff. Given a space X, the expression $C(X)$ stands for the set of all real-valued continuous function on X and $C^*(X) = \{f \in C(X) : f \text{ is bounded}\}$. The set $C(X)$ endowed with the topology of pointwise convergence is denoted by $C_p(X)$ and $C_p^*(X)$ is the set $C^*(X)$ with the topology inherited from $C_p(X)$. It is worth mentioning that both $C_p(X)$ and $C_p^*(X)$ are dense linear subspaces of $\mathbb{R}^X$. The study of spaces $C_p(X)$ is often called C_p-theory; an overview of what has been achieved in this area can be found in the books [11]–[14].

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Gul'ko and Khmyleva proved in [6], among other things, that the space $C_p(\mathbb{R})$ is homeomorphic to $C_p([0,1])$ and hence compactness is not preserved by homeomorphisms of spaces $C_p(X)$. The results of Dobrowolski, Marciszewski and Mogilski [5] imply that there exists a compact space X such that $C_p^*(X)$ is homeomorphic to $C_p^*(Y)$ for some non-compact space Y; here X can be the convergent sequence $\{0\} \cup \{\frac{1}{n+1} : n \in \omega\}$ with its limit and Y the space $\mathbb{Q}$ of rational numbers.

However, if a homeomorphism $\xi : C_p(X) \to C_p(Y)$ is linear, then X is compact if and only if so is Y: this was proved by Arhangel'skii in [1]. In [15] Uspenskij strengthened Arhangel'skii's result showing that even uniform continuity of ξ is sufficient to preserve both pseudocompactness and compactness. The main purpose of this paper is to show that existence of a linear homeomorphism between $C_p^*(X)$ and $C_p^*(Y)$ implies that the space X is pseudocompact if and only if so is Y. An immediate consequence of this result is the fact that compactness is also preserved by linear homeomorphisms between spaces of bounded functions. We will also show that Arhangel'skii's method of proof does not work in our case and neither does Uspenskij's proof.

2. Linear equivalence and pseudocompact spaces. In [1] Arhangel'skii proved the following result:

THEOREM 2.1. *Let X and Y be spaces such that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. Then X is pseudocompact if and only if Y is pseudocompact.*

We will show that this result also holds for linearly homeomorphic function spaces $C_p^*(X)$ and $C_p^*(Y)$. Besides, the method of proof of Theorem 2.1 used in [1] cannot be applied in the case of spaces of bounded functions. To demonstrate this let us outline the proof given in [1].

Assume that X and Y are spaces and let $\phi : C_p(X) \to C_p(Y)$ be a continuous linear function. For $y \in Y$, the map $\psi_y : C_p(X) \to \mathbb{R}$ defined by $\psi_y(f) = \phi(f)(y)$ is continuous and linear. This means that ψ_y belongs to the dual $L(X)$ of the space $C_p(X)$. Since the set of evaluation mappings $\{\xi_x : x \in X\}$ defined by $\xi_x(f) = f(x)$ for all $x \in X$ and $f \in C_p(X)$, form a Hamel basis for $L(X)$, there are $x_1, \ldots, x_n \in X$ and $\lambda_{x_1}^y, \ldots, \lambda_{x_n}^y \in \mathbb{R}$ such that $\psi_y = \sum_{i=1}^n \lambda_{x_i}^y \xi_{x_i}$. Therefore we have the equality $\phi(f)(y) = \sum_{i=1}^n \lambda_{x_i}^y f(x_i)$ for every $f \in C_p(X)$. The set $\mathrm{supp}(y) = \{x_1, \ldots, x_n\}$ will be called the *support* of y. For $B \subset Y$, we denote $\bigcup_{y \in B} \mathrm{supp}(y)$ by $\mathrm{supp}(B)$. Similarly, we can define a support function for continuous linear functions $\phi : C_p^*(X) \to C_p^*(Y)$. We define spaces X and Y to be l_p-equivalent (resp. l_p^*-equivalent) if $C_p(X)$ and $C_p(Y)$ (resp. $C_p^*(X)$ and $C_p^*(Y)$) are linearly homeomorphic.

The main tool for the proof of Theorem 2.1 is the following result in [1]:

LEMMA 2.2. *Given spaces X and Y, let $\phi : C_p(X) \to C_p(Y)$ be a continuous linear function. If $A \subset Y$ is bounded, then $\mathrm{supp}(A) \subset X$ is bounded.*

Theorem 2.1 now easily follows from the fact that if ϕ is a linear homeomorphism, then $X = \mathrm{supp}(Y)$ (see [7, Lemma 6.8.2, p. 406]). To see that compactness is preserved by l_p-equivalence, it suffices to recall the following theorem of Okunev (see [9]).

THEOREM 2.3. *If $C_p(X)$ embeds into $C_p(Y)$ and Y is σ-compact, then X is also σ-compact.*

Therefore, if X is compact and Y is l_p-equivalent to X, then Y is pseudocompact by Theorem 2.1 and σ-compact by Theorem 2.3; this, of course, implies that X is compact.

Let us now try to see where this approach brings us in the case of bounded functions. There is no problem with Okunev's theorem because if $C_p^*(X)$ is homeomorphic to $C_p^*(Y)$, then $C_p(Y)$ embeds in $C_p^*(Y)$ being homeomorphic to $C_p(Y, (0, 1)) = \{f \in C_p(Y) : f(Y) \subset (0, 1)\}$. As a consequence, $C_p(Y)$ embeds into $C_p^*(X) \subset C_p(X)$ so Okunev's theorem is applicable to see that compactness of X implies σ-compactness of Y.

But the crucial Lemma 2.2 does not hold for spaces of bounded functions, as the following example, motivated by Baars and de Groot [3, 1.2.12], shows.

EXAMPLE 2.4. Let $(x_n)_{n \in \mathbb{N}}$ be a faithfully indexed sequence converging to a point $x \notin \{x_n : n \in \mathbb{N}\}$. Consider the space $S = \{x_n : n \in \mathbb{N}\} \cup \{x\}$ and let $Z = S \oplus \omega$ be the topological sum of S and ω.

Define $\phi : C_p^*(Z) \to C_p^*(Z)$ by the equalities

$$\phi(f)(z) = \begin{cases} f(x_n) + \frac{1}{n}f(2n) & \text{if } z = x_n \in S; \\ f(x) & \text{if } z = x \in S; \\ f(2n-1) + f(2n) & \text{if } z = 2n \in \omega; \\ f(2n-1) - f(2n) & \text{if } z = 2n-1 \in \omega. \end{cases}$$

Given any $f \in C_p^*(X)$ and $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $f(Z) \subset [-k, k]$. Choose $m > \frac{2k}{\varepsilon}$ such that $|f(x_m) - f(x)| < \frac{\varepsilon}{2}$ for every $n \geq m$. Then for $n \geq m$,

$$|\phi(f)(x_n) - \phi(f)(x)| = |f(x_n) - f(x) + \frac{1}{n}f(2n)|$$

$$\leq |f(x_n) - f(x)| + |\frac{1}{n}f(2n)| < \frac{\varepsilon}{2} + \frac{k}{n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that $\phi(f)$ is continuous at the unique non-isolated point x and hence ϕ is a well-defined continuous linear function.

Define $\psi : C_p^*(Z) \to C_p^*(Z)$ by

$$\psi(g)(z) = \begin{cases} g(x_n) - \frac{1}{2n}(g(2n) - g(2n-1)) & \text{if } z = x_n \in S; \\ g(x) & \text{if } z = x \in S; \\ \frac{1}{2}(g(2n) - g(2n-1)) & \text{if } z = 2n \in \omega; \\ \frac{1}{2}(g(2n) + g(2n+1)) & \text{if } z = 2n-1 \in \omega. \end{cases}$$

Then   is also a well-defined continuous linear function. Moreover, $\phi \circ \psi$ and $\circ \phi$ are the identity maps on Z. Therefore ϕ is a linear homeomorphism. Note that $\text{supp}(x) = \{x\}$ and we have the equality $\text{supp}(x_n) = \{x_n, 2n\}$ for every $n \in \mathbb{N}$. Hence $\text{supp}(S) = S \cup \{2n : n \in \mathbb{N}\}$. Since S is compact and $\text{supp}(S)$ is unbounded, this shows that Lemma 2.2 indeed does not hold for l_p^*-equivalent spaces.

Observation 2.5. It is even easier to see that, in our case, Uspenskij's method does not work either. Uspenskij established in [15] that a space X is pseudocompact if and only if the uniform space $C_p(X)$ is σ-totally bounded. This trivially implies that, under existence of a uniform homeomorphism between $C_p(X)$ and $C_p(Y)$, the space X is pseudocompact if and only if so is Y. However, the uniform space $C_p^*(X)$ is σ-totally bounded for any space X and therefore pseudocompactness cannot be characterized in this way in terms of the uniform structure of $C_p^*(X)$.

Hence for spaces of bounded functions a new approach is required. We will show that some non-trivial results from Functional Analysis will do the job. Recall that the set $C^*(X)$ endowed with the topology of uniform convergence is a Banach space which we will denote by $C_u^*(X)$. As usual, $\|f\| = \sup_{x \in X} |f(x)|$ is the norm of the function f for any $f \in C^*(X)$. By the closed graph theorem, if $\phi : C_p^*(X) \to C_p^*(Y)$ is a linear homeomorphism then $\phi : C_u^*(X) \to C_u^*(Y)$ is a linear homeomorphism as well. This implies there exists $k \in \mathbb{N}$ such that $\frac{1}{k} \cdot \|f\| \leq \|\phi(f)\| \leq k \cdot \|f\|$ for each $f \in C^*(X)$. Note that if X is compact, then $C^*(X) = C(X)$ and hence we can denote the Banach space $C_u^*(X)$ by $C_u(X)$.

The following result, see, e.g., Semadeni [10, 19.3.1], is a well-known consequence of the Riesz representation theorem and Lebesgue's dominated convergence theorem.

Theorem 2.6. *Given a compact space Z, assume that $(f_n)_{n \in \mathbb{N}}$ is a sequence in $C_u(Z)$ converging pointwise to a function $f \in C_u(Z)$. If $\sup_{n \in \mathbb{N}} \|f_n\| < \infty$, then for every continuous linear functional $T : C_u(Z) \to \mathbb{R}$, we have the equality $\lim_{n \to \infty} T(f_n) = T(f)$.*

We will also need the following result of Okunev [8].

Theorem 2.7. *Given a space X, let υX be its Hewitt realcompactification; observe that the restriction map $\pi : C_p(\upsilon X) \to C_p(X)$ is a bijection. Then the map $\pi|A : A \to \pi(A)$ is a homeomorphism for any countable set $A \subset C_p(X)$.*

We are now ready to present our generalization of Theorem 2.1.

Theorem 2.8. *Suppose that $C_p^*(X)$ and $C_p^*(Y)$ are linearly homeomorphic for some spaces X and Y. Then X is pseudocompact if and only if Y is pseudocompact.*

Proof. Fix a linear homeomorphism $\phi : C_p^*(X) \to C_p^*(Y)$ and assume that Y is pseudocompact and X is not. There exists $k \in \mathbb{N}$ such that $\|\phi(f)\| \leq k \cdot \|f\|$ for every $f \in C_p^*(X)$; denote by βX the Čech-Stone compactification of X.

Observe that every $f \in C_p^*(X)$ has a unique extension $e_X(f) \in C_p(\beta X)$; as an immediate consequence, the space $C_u^*(X)$ is linearly isometric to $C_u(\beta X)$. Analogously, if we denote by $e_Y : C_u(Y) \to C_u(\beta Y)$ the extension map, then e_Y is a linear isometry. Therefore the function $\xi : C_u(\beta X) \to C_u(\beta Y)$ defined by $\xi(f) = e_Y(\phi(f|X))$ for each $f \in C_u(\beta X)$, is a linear homeomorphism, and $\xi^{-1}(h) = e_X(\phi^{-1}(h|Y))$ for every $h \in C_u(\beta Y)$.

The fact that the space X is not pseudocompact implies that there exists a discrete family $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of X; for every $n \in \mathbb{N}$,

pick a point $x_n \in U_n$ and choose a function $f_n : X \to [0, 1]$ such that $f_n(x_n) = 1$ and $f_n(X \setminus U_n) = 0$. Let $p \in \beta X \setminus X$ be a cluster point of the sequence $\{x_n : n \in \mathbb{N}\}$. For every $n \in \mathbb{N}$, consider the functions $g_n = \sum_{m \geq n} f_n$ and $h_n = \phi(g_n)$. It is easy to see that $g_n \in C_u^*(X)$ and $\|g_n\| = 1$ while $g_n \to 0$ pointwise in $C_u^*(X)$. Then $h_n \to 0$ pointwise in $C_u(Y)$ and $\|h_n\| \leq k$ for every $n \in \mathbb{N}$.

Since Y is pseudocompact, the space βY coincides with the Hewitt realcompactification υY of the space Y so we can apply Theorem 2.7 to see that the restriction of the map $e_Y : C_p(Y) \to C_p(\beta Y)$ to any countable subset of $C_p(Y)$ is continuous. In particular, e_Y restricted to the set $\{h_n : n \in \mathbb{N}\} \cup \{0\}$ is continuous and hence the sequence $\{e_Y(h_n) : n \in \mathbb{N}\}$ converges to 0 in the space $C_p(\beta Y)$. Besides, $\|e_Y(h_n)\| = \|h_n\| \leq k$ for every $n \in \mathbb{N}$ because e_Y is an isometry.

Letting $T(f) = f(p)$ for every $f \in C_u(\beta X)$ we obtain a continuous linear functional $T : C_u(\beta X) \to \mathbb{R}$. Observe that $T(e_X(g_n)) = e_X(g_n)(p) = 1$ for every $n \in \mathbb{N}$.

Define $S : C_u(\beta Y) \to \mathbb{R}$ by $S(h) = T(\xi^{-1}(h))$ for every $h \in C_u(Y)$. Then S is a continuous linear functional on $C_u(\beta Y)$ and we have the equalities

$$S(e_Y(h_n)) = T(e_X(\phi^{-1}(h_n))) = T(e_X(g_n)) = 1$$

for every $n \in \mathbb{N}$, while $S(e_Y(h_n)) \to S(0) = 0$ by Theorem 2.6, which is a contradiction. □

COROLLARY 2.9. *Assume that $C_p^*(X)$ and $C_p^*(Y)$ are linearly homeomorphic for some spaces X and Y. Then X is compact if and only if Y is compact.*

Proof. Assume that Y is compact and observe that $C_p(X)$ embeds in $C_p^*(X)$ being homeomorphic to $C_p(X, (0, 1))$. Therefore $C_p(X)$ embeds in $C_p^*(Y)$ which, in turn, embeds in $C_p(Y)$. Therefore $C_p(X)$ embeds in $C_p(Y)$ so X is σ-compact by Theorem 2.3. Since X is also pseudocompact by Theorem 2.8, it must be compact. □

OBSERVATION 2.10. The proof of Corollary 2.9 uses the fact that σ-compactness is preserved by l_p^*-equivalence. Since, in presence of pseudocompactness, even the Lindelöf property implies compactness, it would be sufficient to establish that l_p^*-equivalence preserves the Lindelöf number. Bouziad showed in [4] that if spaces X and Y are l_p-equivalent, then the Lindelöf numbers of X and Y coincide. A careful examination of the proof of this result in [14] shows that also for l_p^*-equivalent spaces X and Y the Lindelöf numbers of X and Y are equal. The proof is literally the same.

QUESTION 2.11. *Suppose that $C_p^*(X)$ and $C_p^*(Y)$ are uniformly homeomorphic. Is it true that X is pseudocompact if and only if so is Y?*

It is an old question, published by Arhangel'skii in 1988 (see [2]) whether countable compactness is preserved by l_p-equivalence; this question was also repeated in [14, Problem 4.4.1]) so, maybe, it is time to find out what happens in the case of l_p^*-equivalence.

QUESTION 2.12. *Suppose that $C_p^*(X)$ and $C_p^*(Y)$ are linearly homeomorphic and X is countably compact. Must Y be countably compact?*

REFERENCES

1. A.V. Arhangel'skii, On linear homeomorphisms of function spaces, *Soviet Math. Doklady* **25** (1982), 852–855.

2. —————— , *Some results and problems in $C_p(X)$-theory,* General Topology and Its Relations to Modern Analysis and Algebra, VI, pp. 11–31, Heldermann Verlag, Berlin, 1988.

3. J. Baars and J. de Groot, *On topological and linear equivalence of certain function spaces,* CWI Tracts, Vol. 86, CWI, Amsterdam, 1992.

4. A. Bouziad, Le degré de Lindelöf est *l*-invariant, *Proc. Amer. Math. Soc.* **129** (2001), 913–919.

5. T. Dobrowolski, W. Marciszewski, and J. Mogilski, On topological classification of function spaces of low Borel complexity, *Trans. Amer. Math. Soc.* **328** (1991), 307–324.

6. S.P. Gul'ko and T.E. Khmyleva, The compactness is not preserved by the relation of *t*-equivalence, *Math. Notes* **39**(6) (1986), 484–488.

7. J. van Mill, *The Infinite-Dimensional Topology of Function Spaces,* Vol. 64, North Holland, Amsterdam, 2002.

8. O.G. Okunev, Spaces of functions in the topology of pointwise convergence: Hewitt extension and τ-continuous functions, *Moscow Univ. Math. Bull.* **40**(4) (1985), 84–87.

9. —————— , Weak topology of an associated space, and *t*-equivalence, *Math. Notes* **46**(1–2) (1990), 534–538.

10. Z. Semadeni, *Banach spaces of continuous functions,* PWN, Warszawa, 1971.

11. V.V. Tkachuk, *A C_p-Theory Problem Book. Topological and Function Spaces,* Springer, New York, 2011.

12. —————— , *A C_p-Theory Problem Book. Special Features of Function Spaces,* Springer, New York, 2014.

13. —————— , *A C_p-Theory Problem Book. Compactness in Function Spaces,* Springer, New York, 2015.

14. —————— , *A C_p-Theory Problem Book. Functional Equivalencies,* Springer, New York, 2016.

15. V.V. Uspenskij, Characterization of compactness in terms of the uniform structure in function spaces, *Russian Math. Surveys* **37**(4) (1982), 143–144.

16. N.V. Velichko, The Lindelöf property is l-invariant, *Topology Appl.* **89** (1998), 277–283.